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FAST TRACK COMMUNICATION

Ultra diffusions

Iddo Eliazar[1] and Joseph Klafter[2,3]

[1] Department of Technology Management, Holon Institute of Technology, PO Box 305, Holon 58102, Israel
[2] School of Chemistry, Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv 69978, Israel
[3] Freiburg Institute for Advanced Studies (FRIAS), University of Freiburg, 79104, Freiburg, Germany

E-mail: eliazar@post.tau.ac.il and klafter@post.tau.ac.il

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Abstract
This communication presents and explores *ultra diffusions*—a class of random transport processes which generalizes the class of 'classic' diffusions. Examples of ultra diffusions include Lévy motions, fractional Brownian motions, fractional stable Lévy motions, Ornstein–Uhlenbeck motions driven by symmetric stable Lévy motions and $M/G/\infty$ processes. A methodological framework of ultra diffusions is established—accommodating transport processes which display, simultaneously, both 'anomalous-diffusion' temporal behavior and 'fat-tailed' amplitudinal Lévy fluctuations. Ultra diffusions with power-law temporal and amplitudinal statistics are shown to emerge universally from a general superposition model of stochastic processes.

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1. Introduction

The most elemental class of random transport processes in science and engineering is the class of diffusions [1]. A key characteristic of diffusions is the linear temporal growth of their mean-square displacements (MSDs). Namely, if $\xi = (\xi(t))_{t\geqslant 0}$ is the trajectory of a diffusion process, then its MSD is given by $\langle \xi(t)^2\rangle = Dt$, where D is the process' diffusion coefficient[4]. An important generalization of diffusions—with wide-ranging applications—is the class of anomalous diffusions [2–5]. This class of processes is characterized by a power-law temporal growth of their MSDs. Namely, if $\xi = (\xi(t))_{t\geqslant 0}$ is the trajectory of an anomalous diffusion process, then $\langle \xi(t)^2\rangle = Dt^{\alpha}$, where D and α are, respectively, the process' anomalous diffusion coefficient and exponent. Anomalous diffusions produce diffusions at

[4] Throughout this communication the notation $\langle\cdot\rangle$ denotes mathematical expectation. Namely, $\langle R\rangle$ is the expectation of a real-valued random variable R.

the exponent value $\alpha = 1$, are termed sub-diffusive in the exponent range $0 < \alpha < 1$, and are termed supper-diffusive in the exponent range $\alpha > 1$.

The generalization of diffusion to anomalous diffusion—using the aforementioned MSD characterization—has a major drawback: it implicitly assumes the existence of a finite variance, thus excluding all transport processes with infinite variance. Transport processes with infinite variance, however, are ubiquitous in various fields of science and engineering [6, 7], and their incorporation into a generalized 'diffusion class' are of both theoretical and experimental importance. In this communication we introduce a class of random transport processes—termed '*ultra diffusions*'—which accommodates both temporal anomalies (i.e. nonlinear MSD growth) and amplitudinal anomalies (i.e. infinite variance).

The mathematical model of diffusion is the Brownian motion $B = (B(t))_{t\geqslant 0}$—the universal scaling limit of random walks. Brownian motion is indeed characterized by its diffusion coefficient D. Alternatively, Brownian motion can also be characterized in Fourier space: $\langle \exp(\mathrm{i}\theta B(t))\rangle = \exp(-(Dt)(\theta^2/2))$ ($-\infty < \theta < \infty$ being the Fourier variable). The Fourier-space characterization of Brownian motion has a special algebraic structure: it decouples into two terms—(Dt) and $(\theta^2/2)$. The term (Dt) is a temporal factor which coincides with the motion's MSD. The term $(\theta^2/2)$ is a Fourier factor which characterizes the motion's Gaussian law. Moreover, the Fourier-space characterization has a major advantage over the MSD characterization: it makes no use of the finite variance of Brownian motion.

Based on the algebraic structure and the 'variance advantage' of the aforementioned Fourier-space characterization of Brownian motion, we propose the following generalization of diffusion. A transport process $\xi = (\xi(t))_{t\geqslant 0}$ is defined to be an *ultra diffusion* if its Fourier-space characterization admits the form

$$\langle \exp(\mathrm{i}\theta\xi(t))\rangle = \exp(-c\cdot\psi(t)\cdot\phi(\theta)), \tag{1}$$

($t \geqslant 0, -\infty < \theta < \infty$), where c is a positive coefficient. The factor $\psi(t)$—henceforth termed the 'temporal function'—is a function depending on the temporal variable t, alone. The factor $\phi(\theta)$—henceforth termed the 'Fourier function'—is a function depending on the Fourier variable θ, alone. In the case of Brownian motion we have $c = D/2$, $\psi(t) = t$ and $\phi(\theta) = \theta^2$. More generally, the temporal function $\psi(t)$ of an ultra-diffusion process—with zero mean and finite variance—coincides, up to a scale factor, with its MSD:

$$\langle \xi(t)^2\rangle = (c\phi''(0))\cdot\psi(t). \tag{2}$$

Hence, the temporal function $\psi(t)$ can indeed be considered as a generalization of the MSD—extending the notion of MSD to ultra-diffusion transport processes with infinite variance.

2. Examples of ultra diffusions

Let us present now three general examples of ultra diffusions. The first general example of ultra diffusion is the class of Lévy processes [8, 9]. This class constitutes of all processes with stationary and independent increments, and has a multitude of applications in science and engineering, including search and animal foraging [10–12], human travel [13, 14] and light scattering [15]. In the case of Lévy processes the temporal function is linear $\psi(t) = t$, and the general form of the Fourier function $\phi(\theta)$ is given by the Lévy–Khinchin formula [8, 9].

Specific examples of Lévy processes include (i) Poisson processes—characterized by the Fourier function $\phi(\theta) = 1 - \exp(\mathrm{i}\theta)$; (ii) compound Poisson processes—i.e. random walks with jump-epochs following a Poisson process—characterized by the Fourier function $\phi(\theta) = 1 - \langle \exp(\mathrm{i}J\theta)\rangle$, where J is a random variable representing the amplitude of the jumps; (iii) Brownian motion—characterized by the Fourier function $\phi(\theta) = \theta^2$; (iv) Cauchy motion—characterized by the Fourier function $\phi(\theta) = |\theta|$. The last two examples—Brownian and Cauchy motions—are the special cases of the sub-class of symmetric stable Lévy motions, whose ultra-diffusion structure is characterized by

$$\psi(t) = t \qquad \text{and} \qquad \phi(\theta) = |\theta|^{\beta}, \tag{3}$$

with exponent β taking values in the range $0 < \beta \leqslant 2$. We note that albeit the special case of Brownian motion ($\beta = 2$), all symmetric stable Lévy motions are of infinite variance. The notion of ultra diffusion thus renders symmetric stable Lévy motions, a natural generalization of diffusion—with a linear 'MSD equivalent', and with 'fat-tailed' probability laws (i.e. $\Pr(|\xi(t)| > l) \approx tl^{-\beta}$ for $l \gg 1$).

The second general example of ultra diffusion is the class of processes which are stochastic integrals driven by symmetric stable Lévy motions. Namely, processes admitting the integral representation

$$\xi(t) = \int_{-\infty}^{\infty} K(t,s)\dot{L}(s)\,\mathrm{d}s, \tag{4}$$

where $K(t,s)$ is a deterministic integration kernel and where $\dot{L}(s)$ is the derivative of a symmetric stable Lévy motion (with exponent $0 < \beta \leqslant 2$). The ultra-diffusion structure of these integral processes is characterized by

$$\psi(t) = \int_{-\infty}^{\infty} |K(t,s)|^{\beta}\,\mathrm{d}s \qquad \text{and} \qquad \phi(\theta) = |\theta|^{\beta} \tag{5}$$

(the Fourier function of these integral processes coincides with the Fourier function of their driving symmetric stable Lévy motions).

Specific examples of these integral processes include the following: (i) Ornstein–Uhlenbeck processes [16–18]—characterized by the temporal function $\psi(t) = 1 - \exp(-r\beta t)$, where $r > 0$ is the corresponding relaxation rate. Ornstein–Uhlenbeck processes are the solutions of the Langevin equation $\dot{\xi} = -r\xi + \dot{L}$ which, in turn, describes one of the most elemental transport dynamics in physics and chemistry [19]. (ii) Fractional Brownian motions [20]—characterized by the temporal function $\psi(t) = t^{2H}$, where $0 < H < 1$ is the motion's Hurst exponent (and $\beta = 2$). Fractional Brownian motions arise in various fields including hydrology [21], DNA sequencing [22], heartbeat dynamics [23] and heat baths [24]. (iii) Fractional stable Lévy motions [25, 26]—characterized by the temporal function $\psi(t) = t^{\beta H}$, where $0 < H < 1$ is the motion's Hurst exponent (and $0 < \beta < 2$). Fractional stable Lévy motions are observed in plasma [27] and in solar wind [28, 29].

The third example of ultra diffusion is the class of $M/G/\infty$ processes. Consider a system to which particles arrive following a Poisson process; the particles are independent of each other, and have i.i.d. lifetimes of random duration L (with finite mean $\langle L \rangle < \infty$). The random process tracking the number of 'alive' particles in the system is refereed to as an $M/G/\infty$ process

$$\xi(t) = \sum_{n=1}^{\infty} \mathbf{I}(A_n \leqslant t < A_n + L_n) \tag{6}$$

($t \geqslant 0$), where A_n and L_n denote, respectively, the arrival epoch and the lifetime of particle n (and $\mathbf{I}(\mathcal{E})$ denotes the indicator function of the event $\mathcal{E}$). The class of $M/G/\infty$ processes originated from the modeling of Geiger–Müller counters—historically referred to as 'type-II counters' [30]—and serves as the most fundamental queueing theory model of infinite-server queueing systems and of infinite-broadband transmission channels [31, 32]. In physics,

Table 1. Examples of ultra diffusions: examples 1–3 are Lévy processes, examples 4–5 are stochastic integrals driven by symmetric stable Lévy motions, and examples 6–7 are M/G/∞ processes.

	Process	Temporal function $\Psi(t) =$	Fourier function $\phi(\theta) =$	Remarks
1	Poisson	t	$1 - \exp(\mathrm{i}\theta)$	
2	Compound Poisson	t	$1 - \langle\exp(\mathrm{i}\theta J)\rangle$	J is the random jump amplitude. Compound Poisson process are CTRWs with exponential waiting times.
3	Symmetric stable Lévy	t	$\lvert\theta\rvert^{\beta}$	$0 < \beta \leqslant 2$ is the Lévy exponent. $\beta = 2$ corresponds to Brownian motion.
4	Ornstein–Uhlenbeck	$1 - \exp(-r\beta t)$	$\lvert\theta\rvert^{\beta}$	$r > 0$ is the relaxation rate. $0 < \beta \leqslant 2$ is the Lévy exponent.
5	Fractional stable Lévy	$t^{\beta H}$	$\lvert\theta\rvert^{\beta}$	$0 < H < 1$ is the Hurst parameter. $0 < \beta \leqslant 2$ is the Lévy exponent. $\beta = 2$ corresponds to fractional Brownian motion.
6	Standard M/G/∞	$\int_0^t \Pr(L > s)\,\mathrm{d}s$	$1 - \exp(\mathrm{i}\theta)$	L is the random particle lifetime.
7	Charged M/G/∞	$\int_0^t \Pr(L > s)\,\mathrm{d}s$	$1 - \langle\exp(\mathrm{i}\theta C)\rangle$	L is the random particle lifetime. C is the random particle charge.

$M/G/\infty$ processes underlie both linear and nonlinear shot noise processes [33]. The ultra-diffusion structure of $M/G/\infty$ processes is characterized by

$$\psi(t) = \int_0^t \Pr(L > s)\,\mathrm{d}s \qquad \text{and} \qquad \phi(\theta) = 1 - \exp(\mathrm{i}\theta) \tag{7}$$

(the Fourier function of $M/G/\infty$ processes coincides with the Fourier function of Poisson processes).

If the particles arriving at the aforementioned $M/G/\infty$ system are charged—the particles having i.i.d. electrical charges of random amplitude C—then the process tracking the system's overall charge is given by

$$\xi(t) = \sum_{n=1}^{\infty} C_n \mathbf{I}\,(A_n \leqslant t < A_n + L_n) \tag{8}$$

($t \geqslant 0$), where C_n denotes the electrical charge of particle n (and where A_n and L_n are as in equation (6)). If the particles' generic lifetime L and charge C are independent random variables, then the ultra-diffusion structure of 'charged $M/G/\infty$ processes' is characterized by

$$\psi(t) = \int_0^t \Pr(L > s)\,\mathrm{d}s \qquad \text{and} \qquad \phi(\theta) = 1 - \exp(\mathrm{i}C\theta) \tag{9}$$

(the Fourier function of 'charged $M/G/\infty$ processes' coincides with the Fourier function of compound Poisson processes).

The aforementioned examples of ultra diffusions are summarized in table 1. We note that ultra diffusions do not encompass the entire class of continuous time random walks (CTRWs) [34]: ultra diffusions include only CTRWs with exponential waiting times—which coincide with the class of compound Poisson processes (discussed above).

3. Universal generation of ultra diffusions

We now turn to explore a general superposition model of stochastic processes which is capable of yielding ultra diffusion in a universal fashion. As in the definition of ultra diffusion, in order to gain intuition we first re-examine the elemental example of Brownian motion.

Brownian motion is a macroscopic manifestation of a microscopic phenomenon. Indeed, as observed by Sir Robert Brown, the jagged and erratic trajectory of a pollen particle suspended in liquid is caused by the 'bombardment effect' of trillions of molecules hitting the particle at random. As a general conceptual model of such a random motion, consider the trajectory of a probe tossed into a turbulent stochastic 'bath'. The probe is constantly impacted by random 'gusts'—these impacts generating the probe's random trajectory $Y = (Y(t))_{t\geqslant 0}$. A fairly general mathematical model for the random probe trajectory Y is the impacts-superposition model

$$Y(t) = \sum_{\tau_n \leqslant t} a_n X_n(\omega_n(t - \tau_n)), \tag{10}$$

where $X_n = (X_n(t))_{t\geqslant 0}$ is the random 'impact pattern' by which gust n affects the probe, and (τ_n, ω_n, a_n) are the random 'impact parameters' of gust n—the gust's initiation epoch $\tau_n \geqslant 0$, and the gust's frequency $\omega_n > 0$ and amplitude $-\infty < a_n < \infty$.

The gusts' impact patterns are assumed to be, statistically, of the same type—implying that the random patterns $\{X_n\}$ are i.i.d. copies of a generic random impact pattern $X = (X(t))_{t\geqslant 0}$, which describes the effect of a single arbitrary gust on the probe's trajectory. The gusts' impact parameters $\mathcal{P} = \{(\tau_n, \omega_n, a_n)\}_n$ form a random collection of points in the three-dimensional domain $[0, \infty) \times (0, \infty) \times (-\infty, \infty)$, and are assumed to be a Poisson process with intensity $\lambda_{\mathcal{P}}(\tau, \omega, a)$ $(\tau \geqslant 0, \omega > 0, -\infty < a < \infty)$ [35].

Poisson processes are the most commonly applied statistical model for the random scattering of points in general domains—with applications ranging from insurance and finance [36] to queueing systems [37]. The informal meaning of the Poissonian intensity $\lambda_{\mathcal{P}}(\tau, \omega, a)$ is as follows: a particle with propagation parameters belonging to the infinitesimal box $(\tau, \tau + \mathrm{d}\tau) \times (\omega, \omega + \mathrm{d}\omega) \times (a, a + \mathrm{d}a)$ exists with the probability $\lambda_{\mathcal{P}}(\tau, \omega, a)\,\mathrm{d}\tau\,\mathrm{d}\omega\,\mathrm{d}a$.

The superposition model of equation (10) was shown to yield—in a universal fashion—anomalous diffusion [38]. The universal generation of anomalous diffusion emanated from the following 'MSD-invariance' question: Is there a class of intensities $\lambda_{\mathcal{P}}(\tau, \omega, a)$ which render the MSD of the probe's trajectory Y invariant, up to a scale factor, with respect to the gusts' impact pattern X? The answer obtained is affirmative, and the resulting MSD admits the universal power-law form

$$\langle Y(t)^2 \rangle = c_X \cdot t^{\alpha}, \tag{11}$$

where c_X is a coefficient depending on the gusts' impact pattern X. Thus, in the context of the superposition model of equation (10), 'MSD-invariance' exclusively yields anomalous diffusion.

In this communication we replace the MSD characterization of transport processes by a Fourier characterization—leading, in turn, to the class of ultra diffusions. In the context of the superposition model of equation (10) the Fourier analog of the aforementioned

'MSD-invariance' question is the following 'Fourier-invariance' question: Is there a class of intensities $\lambda_{\mathcal{P}}(\tau, \omega, a)$ which render the probe's trajectory Y a ultra diffusion—with temporal function $\psi(t)$ and Fourier function $\phi(\theta)$ which are *invariant* with respect to the gusts' impact pattern X?

An analysis based on probabilistic conditioning, and on Campbell's theorem of the theory of Poisson processes ([35], section 3.2), asserts that the answer to the 'Fourier-invariance' question is affirmative: 'Fourier invariance' holds if and only if the intensity $\lambda_{\mathcal{P}}(\tau, \omega, a)$ admits the functional form

$$\lambda_{\mathcal{P}}(\tau, \omega, a) = \varphi(\tau\omega) \cdot \omega^{-\alpha} \cdot |a|^{-1-\beta}, \tag{12}$$

where $\varphi(s)$ ($s \geqslant 0$) is a non-negative valued function, α is a positive exponent and β is an exponent taking values in the range $0 < \beta < 2$. The function $\varphi(s)$ couples together the gusts' initiation epochs and frequencies, and is henceforth termed the 'coupling function'. Different choices of the coupling function $\varphi(s)$ result in different 'impact scenarios'. For example, setting the coupling function $\varphi(s)$ to be a constant function yields a 'steady-state scenario' in which the gusts impact the probe in a time-homogeneous fashion. On the other hand, setting the coupling function $\varphi(s)$ to be Dirac's delta function yields a 'big-bang scenario' in which all gusts impact the probe at time $t = 0$.

A further analysis implies that the intensity of equation (12) yields the Fourier characterization

$$\langle \exp(\mathrm{i}\theta Y(t)) \rangle = \exp(-c_X \cdot t^{\alpha} \cdot |\theta|^{\beta}). \tag{13}$$

Namely, 'Fourier invariance' implies that the universal functional structure of both the temporal and Fourier functions is the power law: $\psi(t) = t^{\alpha}$ ($\alpha > 0$) and $\phi(\theta) = |\theta|^{\beta}$ ($0 < \beta < 2$). On the one hand, the universal temporal function $\psi(t) = t^{\alpha}$ is the 'MSD equivalent' of anomalous diffusion. On the other hand, the universal Fourier function $\phi(\theta) = |\theta|^{\beta}$ characterizes the infinite-variance symmetric stable Lévy laws ($0 < \beta < 2$). 'Fourier invariance' hence yields the simultaneous display of both temporal and amplitudinal anomalous statistics.

The Fourier characterization of equation (13) coincides with the ultra-diffusion structure of fractional stable Lévy motions with the Hurst exponent $H = \alpha/\beta$. Yet, this does not imply that the probe's trajectory Y—generated by intensities admitting the functional form of equation (12)—is a fractional stable Lévy motion. Indeed, the definition of ultra diffusion is based on the Fourier characterization of the random variables $Y(t)$ ($t \geqslant 0$), and does not characterize the entire stochastic process $Y = (Y(t))_{t \geqslant 0}$. Namely, different stochastic processes can admit the Fourier characterization of equation (13)—yet there is only one fractional stable Lévy motion with the Hurst exponent $H = \alpha/\beta$ and Lévy exponent β.

The Fourier characterization of equation (13) implies that the probe's trajectory Y—generated by intensities admitting the functional form of equation (12)—has an intrinsic self-similar structure: for all $t \geqslant 0$ we have

$$Y(t) \overset{\text{Law}}{=} t^{\alpha/\beta} \cdot Y(1) \tag{14}$$

(the equality sign representing equality in law). Namely, the random variable $Y(t)$ is equal, in law, to the random variable $Y(1)$ multiplied by the scale factor $t^{\alpha/\beta}$. The intrinsic self-similar structure of equation (14) is a weak form of statistical self-similarity [39]—displayed by symmetric stable Lévy motions, fractional Brownian motions and fractional stable Lévy motions—in which the processes' trajectories are statistically invariant under 'zoom-in' and 'zoom-out' operations. The universal generation of statistical self-similarity, in the context of the stochastic superposition model of equation (10), was explored in [40].

The value of the coefficient c_X appearing in equation (13) is given by

$$c_X = \int_0^\infty \int_{-\infty}^\infty (1 - \langle \exp(\mathrm{i}aX(\omega))\rangle) \frac{\tilde{\varphi}_{1+\alpha}(\omega)}{|a|^{1+\beta}} \,\mathrm{d}\omega\,\mathrm{d}a, \tag{15}$$

where $\tilde{\varphi}_{1+\alpha}(\omega) = \int_0^\infty \varphi(s)(\omega+s)^{-1-\alpha}\,\mathrm{d}s$ $(\omega > 0)$. The integral appearing on the right-hand side of equation (15) is required to converge. Hence, given a coupling function $\varphi(s)$ equation (15) prescribes the integrability condition that need be met by the impact pattern X. Examples of coupling functions $\varphi(s)$ include (i) the constant coupling function $\varphi(s) = 1$—representing the 'steady-state scenario' in which the gusts impact the probe in a time-homogeneous fashion—yielding $\tilde{\varphi}_{1+\alpha}(\omega) = \omega^{-\alpha}$; (ii) the Dirac coupling function $\varphi(s) = \delta(s)$—representing the 'big-bang scenario' in which all gusts impact the probe at time $t = 0$—yielding $\tilde{\varphi}_{1+\alpha}(\omega) = \omega^{-1-\alpha}$; (iii) the Heaviside coupling function $\varphi(s) = \mathbf{I}(s > 1)$—yielding $\tilde{\varphi}_{1+\alpha}(\omega) = (1+\omega)^{-\alpha}$.

4. Conclusions

In this communication we presented and explored the class of *ultra diffusions*, which generalizes the class of 'classic' diffusions—accommodating transport processes displaying both temporal and amplitudinal anomalous statistics. Counterwise to the class of anomalous diffusions—which is based on the MSD characterization of transport processes, and hence implicitly imposing a finite-variance condition—the class of ultra diffusions is based on a Fourier characterization which does not require finite variance. Consequently, the class of ultra diffusions engulfs a wide array of transport processes which are natural generalizations of 'classic' diffusions. Examples of ultra diffusions include: Lévy motions, fractional Brownian motions, fractional stable Lévy motions, Ornstein–Uhlenbeck motions driven by symmetric stable Lévy motions and $M/G/\infty$ processes.

Ultra-diffusion transport processes are characterized by two functions: a 'temporal function' $\psi(t)$ which quantifies the transport's temporal dispersion, and a 'Fourier function' $\phi(\theta)$ which characterizes the transport's underlying probability law. The temporal function was shown to be a generalization of the MSD function—which coincides, up to a scale factor, with the MSD in the case of finite-variance transport processes.

In the second part of the communication we considered a general stochastic superposition model which is capable of generating, in a universal fashion, ultra diffusions. The superposition model describes the random motion of a probe tossed into a stochastic bath. The probe's trajectory Y is the superposition of the effects of all gusts impacting it. The gusts are i.i.d. and share a statistically common impact pattern X representing the gusts–probe interaction, and each gust has its own random impact parameters—initiation epoch, frequency and amplitude. Our aim was to characterize the Poisson statistics of the impact parameters $\mathcal{P}$ which render the probe's motion a universal ultra diffusion: characterize the class of intensities $\lambda_{\mathcal{P}}(\tau, \omega, a)$ for which the probe's trajectory Y is a ultra diffusion—with temporal function $\psi(t)$ and Fourier function $\phi(\theta)$ which are *invariant* with respect to the gusts' impact pattern X. The corresponding 'universal' temporal and Fourier functions turned out to be power laws: the temporal power law $\psi(t) = t^\alpha$ $(\alpha > 0)$ implying a generalization of anomalous diffusion and the Fourier power law $\phi(\theta) = |\theta|^\beta$ $(0 < \beta < 2)$ implying infinite-variance symmetric stable Lévy laws. Hence—in the context of the aforementioned stochastic superposition model—the 'universal' ultra-diffusion statistics display, simultaneously, both temporal and amplitudinal anomalies.

This communication provides theoretical and experimental physicists a methodological approach capable of quantifying and classifying a wide array of transport processes via a single and unified ultra-diffusion framework. The ultra-diffusion framework is particularly suited to model transport processes exhibiting both 'anomalous diffusion' temporal behavior and 'fat-tailed' amplitudinal Lévy fluctuations.

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